ON THE THEORY OF DYNAMIC PROGRAMMING

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1. Introduction.—We are interested in a class of mathematical problems which arise in connection with situations which require that a bounded or unbounded sequence of operations be performed for the purpose of achieving a desired result. Particularly important are the cases where each operation gives rise to a stochastic event, the result of which is applied to the determination of subsequent operations.

Two fundamental problems encountered in situations of this type, in some sense duals of each other, are those of maximizing the yield obtained in a given time, or of minimizing the time or cost required to accomplish a certain task.

In many cases, the problem of determining an optimal sequence of operations may be reduced to that of determining an optimal first operation. The general class of functional equations generated by problems of this nature has the form

$$f(p) = \begin{Bmatrix} \text{min.} \\ \text{max.} \\ k \end{Bmatrix} (T_k(f)), \tag{1.1}$$

where T_k is an operator. In many cases of interest, the operator has the form

$$T_k(f) = g_k(p) + h_k(p)f(S_k p), \tag{1.2}$$

where S_k is a point transformation.

We shall first present some existence and uniqueness theorems pertaining to the solutions of (1.1), and then present explicit solutions of some simple functional equations of the form of (1.1).

As simple examples of problems which give rise to functional equations of this form, we mention the following:

1. We are given the fact that one of N boxes contains a ball, with probability p_k that the ball is in the kth box. Let q_k be the probability that on examining the kth box we are unable to examine its contents, and t_k be the time consumed in one examination. What procedure minimizes the expected time required to locate the box containing the ball, and what procedure minimizes the expected time required to obtain the ball?

2. We are given a quantity $x > 0$ which may be divided into two parts, y and $x - y$. From y we obtain a return of $g(y)$ and from $x - y$ a return of $h(x - y)$. In so doing we are left with a new quantity $ay + b(x - y)$, $0 < a, b < 1$, with which to continue the process. How does one pro-

ceed in order to maximize the total return obtained in a finite, or unbounded, number of stages?

The theory of dynamic programming is intimately related to the theory of sequential analysis due to Wald.[3] Two papers by Arrow, Blackwell and Cirshick,[1] and Arrow, Harris and Marschak[2] also treat problems of similar type.

2. *Existence and Uniqueness Theorems.*

THEOREM 1. *Consider the equation*†

$$f(p) = \max_{1 \leq k \leq n}. (g_k(p) + h_k(p)f(S_k p)), \qquad p \in R, \tag{2.1}$$

where we assume that

$$\begin{aligned} &(a) \quad \textit{If } p \in R, \textit{ a region of } n\textit{-dimensional space, then } S_k p \in R. \quad (2.2)\\ &(b) \quad |g_k(p)| \leq c_1 \quad \textit{for } p \in R,\\ &(c) \quad |h_k(p)| \leq c_2 < 1 \quad \textit{for } p \in R,\\ &(d) \quad g_k(p), h_k(p) \geq 0 \quad \textit{for } p \in R. \end{aligned}$$

Under these assumptions there is a unique bounded solution to (2.1).

THEOREM 2. *Consider the equation*

$$f(x) = \max_{R}. [a(x_1, x_2, \ldots, x_N) + f(b(x_1, x_2, \ldots, x_N))], \tag{2.3}$$

where $R = R(x)$ *is defined by* $x_k \geq 0, \sum_{k=1}^{N} x_k = x.$

If

(a) $a(x_1, x_2, \ldots, x_N)$ *is continuous over* $R(x)$ *for* $0 \leq x \leq x_0$, *non-negative, and* $a(0, 0, \ldots, 0) = 0,$ (2.4)

(b) $b(x_1, x_2, \ldots, x_N)$ *is continuous and non-negative over* R,

(c) $b(x_1, x_2, \ldots, x_N) \leq cx, 0 < c < 1,$ *in* $R(x)$,

(d) $\sum_{l=0}^{\infty} h(c^l x_0) < \infty$, *where* $h(x) = \max_{R}. a(x_1, x_2, \ldots, x_N),$

there is a unique continuous solution to (2.3) for which $f(0) = 0$ *for* $0 \leq x \leq x_0$.

THEOREM 3. *Consider the equation*

$$\begin{aligned} f(p) &= \min.\begin{Bmatrix} 1 + \sum_{k=0}^{N} p_k f(x_k), \\ 1 + f(S_l p) \end{Bmatrix}, \qquad p \neq x_0, \\ f(x_0) &= 0, \end{aligned} \tag{2.5}$$

where $l = 1, 2, \ldots, M$, *and*

$$p = \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_N \end{pmatrix}, \quad S_l p = \begin{pmatrix} p_{0l} \\ p_{1l} \\ \vdots \\ p_{Nl} \end{pmatrix}, \quad x_k = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \tag{2.6}$$

the 1 occurring in the kth place. Each p and $S_l p$ is a probability vector, $p_k \geq 0$, $\sum_{k=0}^{N} p_k = 1$, *and $f(p)$ is a scalar function of p.*

If for each l it is true that

$$\sum_{k=1}^{N} p_{kl} \leq c_1 \sum_{k=1}^{N} p_k, \quad 0 < c_1 < 1, \tag{2.7}$$

there is a unique bounded positive solution to (2.5).

The proof in all three cases employs the method of successive approximations. The equation in (2.5) occurs in connection with problems similar to problem 1 above.

3. *Solutions of Some Particular Functional Equations.*—In this section we indicate the solution of some simple cases of the general equations discussed above.

THEOREM 4. *The solution of*

$$f(x, y) = \max.\begin{bmatrix} p_1[r_1x + f(s_1x, y)] \\ p_2[r_2y + f(x, s_2y)] \end{bmatrix}, \quad x, y \geq 0 \tag{3.1}$$

where $0 < p_1, p_2, s_1, s_2 < 1$, $r_1, r_2 > 0$, is given by

$$\begin{aligned} f(x, y) &= p_1[r_1x + f(s_1x, y)] \quad \text{for} \quad \frac{p_1r_1x}{1 - p_1} \geq \frac{p_2r_2y}{1 - p_2}, \\ &= p_2[r_2y + f(x, s_2y)] \quad \text{for} \quad \frac{p_1r_1x}{1 - p_1} \leq \frac{p_2r_2y}{1 - p_2}. \end{aligned} \tag{3.2}$$

If $s_1^m = s_2^n$, $f(x, y)$ is piecewise linear.

This result may be extended in many ways.

THEOREM 5. *The solution of*

$$f(x) = \max_{0 \leq y \leq x}. [g(y) + h(x - y) + f(ay + b(x - y))], \tag{3.3}$$

where $0 < a, b < 1$, may be reduced to that of

$$f(x) = \max. [g(x) + f(ax), h(x) + f(bx)], \tag{3.4}$$

in $0 \leq x \leq x_0$, if g and h are monotonically increasing functions such that $g(0) = h(0) = 0$, g'', $h'' \geq 0$ in $[0, x_0]$.

If g'', $h'' \leq 0$ the situation is much more complicated, and no such simple result such as (3.4) holds in general. The solution of (3.4) may be obtained explicitly and is similar in structure to that of (3.1) above. This functional equation arises from problem 2.

THEOREM 6. *The solution of*

$$f(p_1, p_2, \ldots, p_N) = \min_k. \left[\frac{t_k}{1 - q_k} + (1 - p_k)f\left(\frac{p_1}{1 - p_k}, \ldots, 0, \ldots, \frac{p_N}{1 - p_k}\right)\right] \tag{3.5}$$

the zero occurring in the kth place, where

$$f(0, \ldots, 0, p_k, \ldots, 0) = \frac{t_k}{1 - q_k}, \tag{3.6}$$

for $p_k > 0$, $k = 1, 2, \ldots, N$, *is given by*

$$f(p_1, p_2, \ldots, p_N) = \frac{t_k}{1 - q_k} + (1 - p_k)f \times \left(\frac{p_1}{1 - p_k}, \ldots, 0, \ldots, \frac{p_N}{1 - p_k}\right) \tag{3.7}$$

if k is the index for which $p_l(1 - q_l)/t_l$ *is a maximum.*

This is the solution to problem 1 above in the case where we wish to obtain the ball. If we want merely to locate the ball the solution is more complicated. In this case we either examine the box for which $p_k(1 - q_k)/t_k$ is a maximum first, or we never examine it.

THEOREM 7. *The solution of*‡

$$f(x) = 1 + \min.\begin{Bmatrix} xf(1) \\ f(ax) \end{Bmatrix}, \quad 1 \geq x > 0, 0 < a < 1, \tag{3.8}$$
$$f(0) = 0,$$

is

$$\begin{aligned} f(x) &= 1 + xf(1), & x \leq x_0 \\ &= 1 + f(ax), & x \geq x_0, \end{aligned} \tag{3.9}$$

where $x_0 = (1 - a^k)/(k + 1)(1 - a)$, *and k is the integer at which* $(y + 1)/(1 - a^y)$ *is a minimum for* $y = 1, 2, \ldots$.

Detailed proofs and further results will appear in another publication.

† Results on the existence of solutions of (2.1) were obtained by S. Karlin and H. N. Shapiro in an unpublished paper.

‡ The solution of (3.1) was obtained in conjunction with M. Shiffman, while that of (3.8) was obtained in conjunction with D. Blackwell.

[1] Arrow, K. J., Blackwell, D., and Girshick, M. A., "Bayes and Minimax Solutions of Sequential Decision Problems," *Econometrica*, **17**, 214–244 (1949).

[2] Arrow, K. J., Harris, T. E., and Marschak, J., Optimal Inventory Policy, Cowles Commission Papers, New Series, No. 44, 1951.

[3] Wald, A., *Statistical Decision Functions*, John Wiley & Sons, New York, 1950.